CHEBYSHEV APPROXIMATIONS

by

PHILLIP F. UNRUH

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Approved by:

S. Thomas Parker
Major Professor

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INTRODUCTION

The purpose of this report is the development of some of the properties and applications of Chebyshev polynomials. This class of functions was named after the Russian mathematician, P. F. Chebyshev (1821-1894). Many papers and books dealing with one or more characteristics of Chebyshev polynomials have been written.

Chebyshev polynomials can be obtained in a variety of ways. These polynomials include the eigenvalue-eigenfunction solutions of a certain differential equation, generating functions, Rodrigues-type formula, recursion formula, trigonometric definition, orthogonality definition, etc. Some of these derivations are discussed in this report. The various definitions lead to the same set of functions, to within a multiplicative constant. Whenever we need to prove some relation, we use that definition or property which is most suitable and most easily applied.

The two properties of Chebyshev polynomials which have spurred their utilization are their Fourier Series property (least-squares approximation) and their minimax property. Both of these are discussed in the report. In addition, they also enter into considerations of continued fraction approximation of functions. Truncated continued fractions lead to Padé approximants, the use of is generally considered to be a very efficient approximative method.

PROPERTIES OF CHEBYSHEV POLYNOMIALS

Many techniques exist for finding polynomial approximations which reduce the amount of work and guarantee a specified accuracy. The Chebyshev polynomials form a set which are among the simplest and most useful of all polynomial approximations.

The Chebyshev polynomial of degree n can be defined by

$$T_n(x) = \cos(n \arccos x).$$

If we set $\theta = \arccos x$, then $T_n(x) = \cos n\theta$. By using de Moivre's theorem,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$$

and noting that

$$\sin \theta = (1 - \cos^2\theta)^{\frac{1}{2}} = (1 - x^2)^{\frac{1}{2}},$$

we have

$$\left[x - i(1 - x^2)^{\frac{1}{2}}\right]^n$$

$$= x^n + nx^{n-1}\left[i(1-x^2)^{\frac{1}{2}}\right] + \frac{n(n-1)}{2!}x^{n-2}\left[i(1-x^2)^{\frac{1}{2}}\right]^2 + \dots$$

Then, by simplifying and taking the real part only of the expansion, we obtain

$$T_n(x) = x^n + \frac{n(n-1)}{2!}x^{n-2}(x^2-1) + \frac{n(n-1)(n-2)(n-3)}{4!}x^{n-4}(x^2-1)^2 + \dots$$

From $T_n(x) = \cos(n \arccos x) = \cos n\theta$, we can develop the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Since $x = \cos \theta$, $-1 \leq x \leq 1$, and $-\pi \leq \theta \leq \pi$ we have that

$$T_{n+1}(x) = \cos(n+1)\theta = \cos n\theta \cos \theta - \sin n\theta \sin \theta$$

and

$$T_{n-1}(x) = \cos(n-1)\theta = \cos n\theta \cos \theta + \sin n\theta \sin \theta$$

Adding, yields

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos n\theta \cos \theta = 2x\, T_n(x);$$

hence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

We can tabulate the Chebyshev polynomials from

$$T_n(x) = \cos n\theta$$

yielding

$$T_o(x) = \cos o = 1$$

and

$$T_1(x) = \cos\theta = x, \text{ for } n = 0,1.$$

From this point all other $T_n(x)$ can be tabulated from the recurrence relation yielding:

$$T_2(x) = 2x^2 - 1$$

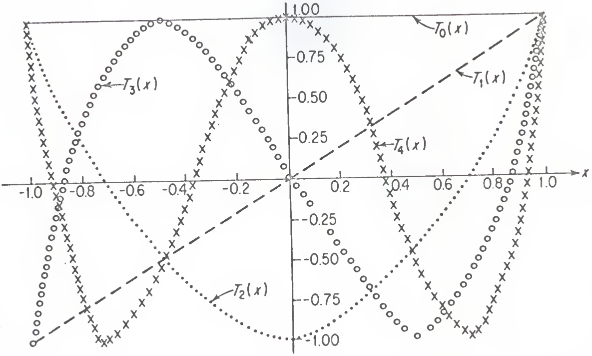
$$T_3(x) = 4x^3 - 3x$$

TABLE A $$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$\vdots$$

The figure below shows the graphs of the first few Chebyshev polynomials.



The Chebyshev polynomials can be shown to satisfy the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + n^2y = 0$$

This equation can be modified so as to conform with the Sturm-Liouville form of the differential equation

$$\frac{d}{dx}\left[r(x)\frac{d\ y}{dx}\right] - \left[q(x) + \lambda\, p(x)\right] \quad y = 0$$

on some interval $a \leq x \leq b$, satisfying certain boundary conditions by dividing by $(1 - x^2)^{\frac{1}{2}}$.

In fitting the differential equation to the Sturm-Liouville second-order differential equation above we note that

$$r(x) = \sqrt{1 - x^2},\ q(x) = 0,\ \ \lambda = n^2,\ p(x) = \frac{1}{\sqrt{1-x^2}},\ \text{and}$$

the range is $[-1, 1]$.

The equation takes the form

$$\frac{d}{dx}\left[\sqrt{1 - x^2}\frac{dy}{dx}\right] - \frac{n^2}{\sqrt{1-x^2}}\,y = 0.$$

We note that for $x = \pm 1$, $r(x) = 0$. Hence this problem has the trivial solution $y = 0$ for any value of the parameter $\lambda = n^2$.

Solutions y, not identically zero, are called characteristic functions or eigenfunctions of the problem, and the values of λ for which such solutions exist, are called characteristic values or eigenvalues of the problem.

Therefore [(1) pp. 44-45] the solutions corresponding to different values of n form an orthogonal set over $[-1, 1]$ with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$.

To solve the differential equation, we can assume a power series development, for each n = 0, 1, 2, . ., we obtain both a polynomial and an infinite series, the polynomial solutions, except for constant factors, are Chebyshev polynomials $y = T_n(x)$ for n = 0, 1, 2, ...

The Chebyshev polynomials also can be shown to be identical with the functions obtained from Rodrigues formula [(6), pp. 15-17].

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \sqrt{1 - x^2} \frac{d^n}{dx^n} \left[(1 - x^2)^{n-\frac{1}{2}}\right]$$

A generating function for the Chebyshev polynomial is

$$\frac{1 - xu}{1 - 2xu + u^2} = \sum_{n=0}^{\infty} T_n(x) u^n$$

This generating function is derived from the geometric series

$$\sum_{n=0}^{\infty} u^n e^{in\theta} = \frac{1}{1 - ue^{i\theta}}, \quad |ue^{i\theta}| \leq 1,$$

by noting that

$$\frac{1}{1 - ue^{i\theta}} = \frac{1}{1 - u\cos\theta - iu\sin\theta}$$

$$= \frac{1 - u\cos\theta + iu\sin\theta}{(1 - u\cos\theta)^2 + u^2\sin^2\theta} .$$

Taking the real parts yields

$$\sum_{n=0}^{\infty} u^n \cos n\theta = \frac{1 - u\cos\theta}{1 - 2u\cos\theta + u^2}$$

Now, the standard substitution of $x = \cos\theta$ shows that the preceding function does generate the Chebyshev polynomials.

From the polynomials preceding, it is easily seen that the leading term of $T_n(x)$ for $n \geq 1$ is $2^{n-1}x^n$. Chebyshev polynomials, $T_n(x)$, have simple zeros at the n points $x_k = \cos\frac{2k-1}{2n}\pi$; $k = 1, 2, \ldots n$. This is seen by substituting x_k in $T_n(x_k) = \cos(n \arccos x)$. Hence:

$$T_n(x_k) = \cos\left[n \arccos\left(\cos\frac{2k-1}{2n}\pi\right)\right] = \cos\frac{2k-1}{2}\pi = 0$$

for $k = 1, 2, \ldots n$.

Then $T_n'(x) = -\sin(n \arccos x)\frac{-n}{\sqrt{1-x^2}} = \frac{n}{\sqrt{1-x^2}}\sin(n \arccos x)$.

Therefore $T_n'(x_k) = \frac{n}{\sqrt{1-x^2}}\sin\frac{2k-1}{2}\pi \neq 0$ implies that the zeros must be simple. This can also be seen on observing that the degree of $T_n(x)$ is n; therefore, $T_n(x)$ has exactly n zeros. Also the zeros must be simple since $T_n(x)$ has n different zeros.

Another useful property of $T_n(x)$ is that its extreme values are ± 1. These $n + 1$ values are obtained for $x_k' = \cos\frac{k}{n}\pi$. From the proof above, $T_n'(x_k') = n(1-\cos\frac{k}{n})^{-\frac{1}{2}}\sin k\pi = 0$ for $k = 1, 2, \ldots n-1$. Then $T_n(x_k') = \cos\left[n \arccos(\cos\frac{k}{n}\pi)\right] = \cos k\pi = (-1)^k$ for $k = 0, 1, \ldots n$. But since $-1 \leq x \leq 1$, $T_n(x) = \cos(n \arccos x)$ implies that $|T_n(x)| \leq 1$. Hence this shows that the values x_k' give extreme values.

Since $T_n(x)$ is a polynomial of degree n in x, it can be shown that if n is even, $T_n(x)$ is an even polynomial; and if n is odd, $T_n(x)$ is an odd polynomial. This property follows from the definition of odd and even functions. An odd function is defined as one in which f(x) is given by $f(-x) = -f(x)$. An even function is one in which f(x) is given by $f(-x) = f(x)$.

The parity of $T_n(x)$ is obtained from the power series expansion given at the beginning of this section. One notes that $(x^2 - 1)$ occurs to different powers while the multipliers of $(x^2 - 1)^k$ are x^n, x^{n-2}, x^{n-4}, ... Additionally, the parity can also be seen from the following argument: let

$$\theta = \text{arc cos } x$$

then $$\text{arc cos } (-x) = \pi - \theta;$$

$$\begin{aligned}
T_n(-x) &= \cos\left[n \text{ arc cos } (-x)\right] \\
&= \cos\left[n(\pi - \theta)\right] \\
&= \cos\left[(n\pi - n\theta)\right] \\
&= \cos n\pi \cos n\theta + \sin n\pi \sin n\theta \\
&= (-1)^n \cos n\theta \\
&= (-1)^n \cos(n \text{ arc cos } x) \\
&= (-1)^n\, T_n(x).
\end{aligned}$$

The Chebyshev polynomials have the properties of both trigonometric functions and orthogonal polynomials. The trigonometric functions have the properties that alternating maxima and minima of individual terms are equal in size, and that the sine and cosine are orthogonal over both the continuous range and an equally spaced set of discrete data points. Orthogonal polynomials have the additional properties that their zeros interlace each other; they can be put into a power series form; and they satisfy a three term recurrence formula.

It can be shown that the Chebyshev polynomials are orthogonal over $[-1, 1]$ with respect to the weighting factor $(1 - x^2)^{-\frac{1}{2}}$. Recall the trigonometric integral

$$\int_0^{\pi} \cos n\theta \cos m\theta \, d\theta = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

This is sometimes referred to as the Fourier expression for orthogonality of the cosine functions. Then with the change of variable $x = \cos\theta$ the above becomes

$$\int_{-1}^{1} \frac{T_n(x) T_m(x) dx}{\sqrt{1 - x^2}} = C_n \delta_{mn},$$

where $C_0 = \pi$, $C_n = \frac{\pi}{2}$ $(n \neq 0)$ and δ_{mn} is the Kronecker delta symbol defined by

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Hence the Chebyshev polynomials are orthogonal over the interval $[-1, 1]$ with respect to the weighting function $w(x) = \frac{1}{\sqrt{1-x^2}}$

Correspondingly, in the discrete domain for N equally spaced data points, we get the relations

$$\sum_{k=0}^{N-1} \cos m\theta_k \cos n\theta_k = \begin{cases} 0 & m \neq n \\ \frac{N}{2} & m = n \neq 0 \\ N & m = n = 0 \end{cases}$$

Hence,

$$(1) \qquad \sum_{k=0}^{N-1} T_m(x_k')T_n(x_k') = \begin{cases} 0 & m \neq n \\ \frac{N}{2} & m = n \neq 0 \\ N & m = n = 0 \end{cases}$$

where $x_k' = \cos \frac{2k+1}{2n}\pi$, $k = 0, 1, \ldots n-1$.

Chebyshev showed that, of all monic polynomials of degree n, the polynomial $T_n(x)2^{1-n}$ has the smallest least upper bound for its absolute value in the interval $-1 \leq x \leq 1$. Since the upper bound of $|T_n(x)|$ is 1, the upper bound inquestion is $\frac{1}{2^{n-1}}$. This property is known as the Chebyshev criterion and will be proved later. The "Chebyshev approximation" is associated with those approximations which attempt to reduce the maximum error to a minimum. This is known as the minimax principle. Normal least-squares approximation minimizes the average square error, but could allow isolated extreme errors. Chebyshev approximation keeps extreme errors down but allows a larger average square error.

N^{th} DEGREE LEAST-SQUARE POLYNOMIAL APPROXIMATION

The n-th degree least-squares polynomial approximation to $f(x)$ in $[-1, 1]$ relative to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ is defined by

$$y(x) = \sum_{k=0}^{n} a_k T_k(x) \qquad -1 \leq x \leq 1$$

$$\text{where} \quad a_o = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)dx}{\sqrt{1-x^2}}$$

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)\ T_k(x)dx}{\sqrt{1-x^2}} \qquad k \neq 0$$

It has the property that, of all polynomials of degree n or less, the integrated weighted squared error

$$\int_{-1}^{1} \frac{[f(x) - y_n(x)]^2 dx}{\sqrt{1-x^2}}$$

is least when $y_n(x)$ is given by

$$\sum_{k=0}^{n} a_k T_k(x).$$

In practice the above formulas are seldom used to evaluate a_k, but they do yield the following upper bounds:

$$|a_o| \leq 2M$$

$$|a_k| \leq \frac{2M}{\pi} \int_0^{\pi} \cos k\theta \, d\theta = \frac{4M}{\pi} \qquad k > 0$$

where M is the maximum value of $|f(x)|$ in $[-1, 1]$.

A more useful method of evaluating the coefficients is based on relations of (1). The functions $T_0(x)$, $T_1(x)$, . . . are orthogonal under integration over $[-1, 1]$ relative to $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Now the same functions $T_0(x)$, $T_1(x)$, . . . $T_n(x)$ are orthogonal under summation over the zeros of $T_{n-1}(x)$ relative to $w(x) = 1$. The n-th degree Chebyshev polynomial approximation to f(x) is defined by

$$f(x) = \sum_{k=0}^{n}{}' a_k T_k(x)$$

where $\sum'$ indicates a finite sum whose first term is to be halved and

$$a_k = \frac{2}{n+1} \sum_{k=0}^{n} f(x_j) T_k(x_j)$$

and where $x_j = -\cos \frac{2j+1}{2n+2}\pi$, $\quad j = 0, 1, . . . n.$

As an example of the above process, suppose that we want to find the second degree Chebyshev approximation through the following points:

$$x_o = -\cos\frac{\pi}{6}, \qquad x_1 = -\cos\frac{\pi}{2}, \qquad \text{and } x_2 = -\cos\frac{5\pi}{6}.$$

$$f(x_o) = \frac{-10\sqrt{3}-3}{4}, \qquad f(x_1) = 0, \qquad f(x_3) = \frac{10\sqrt{3}-3}{4}.$$

We can tabulate the following data:

	x_j	$T_o(x_j)$	$T_1(x_j)$	$T_2(x_j)$	$f(x_j)$
$x = -\cos\frac{\pi}{6}$	$\frac{-\sqrt{3}}{2}$	1	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{-10\sqrt{3}-3}{4}$
$x = -\cos\frac{\pi}{2}$	0	1	0	-1	0
$x = -\cos\frac{5\pi}{6}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{10\sqrt{3}-3}{4}$

Then for the coefficients we calculate

$$a_o = \frac{2}{3}\sum_{j=0}^{2} f(x_j)T_o(x_j) = -1,$$

$$a_1 = \frac{2}{3}\sum_{j=0}^{2} f(x_j)T_1(x_j) = 5,$$

$$a_2 = \frac{2}{3}\sum_{j=0}^{2} f(x_j)T_2(x_j) = -\frac{1}{2}$$

Therefore,

$$f(x) = {\sum_{k=0}^{2}}' a_k T_k(x)$$

$$= \frac{a_o}{2} + a_1T_1(x) + a_2T_2(x)$$

$$= -\tfrac{1}{2} + 5x - \tfrac{1}{2}(2x^2-1)$$

$$= 5x - x^2$$

In the above note that the $T_1(x)$ satisfy the orthogonality condition (1) and also that the x_j are the zeros of $T_3(x)$.

THE MINIMAX PRINCIPLE

We shall now restate Chebyshev's theorem, or the "minimax" principle. It states that of all polynomials $P_n(x)$ of degree n having a leading coefficient equal to 1, the polynomial

$$\frac{T_n(x)}{2^{n-1}}$$

has the least least-upper bound for its absolute value in the interval $[-1, 1]$; i.e.,

$$\operatorname*{Max}_{-1\le x\le 1}\left|\frac{T_n(x)}{2^{n-1}}\right| \le \operatorname*{Max}_{-1\le x\le 1}\left|P_n(x)\right|$$

To prove this we shall assume that there is a monic polynomial, say $P_n(x)$ of degree n, for which the max $P_n(x)$ has a lesser absolute value. Now since the coefficient of x^n in $T_n(x)$ is 2^{n-1} the leading term of $2^{1-n}T_n(x)$ is x^n. Hence we can form the polynomial

$$Q(x) = 2^{1-n}T_n(x) - P_n(x)$$

where $Q(x)$ is of degree n-1 or less. Now, since $P_n(x)$ has a smaller norm than $\frac{T_n(x)}{2^{n-1}}$, $Q(x)$ must be positive at the maxima of $T_n(x)$ and negative at the corresponding minimia of $T_n(x)$. Since $\frac{T_n(x)}{2^{n-1}}$ takes on extreme values $n + 1$ times at the points

$x_j = \cos\frac{j\pi}{n}$, $j = 0, 1, \ldots n$, in the interval $[-1, 1]$, $Q(x)$ must vanish at least n times. But as observed from above, $Q(x)$ is a polynomial of degree at most n-1. Thus a contradiction has been shown and the theorem is proved.

This theorem may be stated in the equivalent way: Among all polynomials of degree n with maximum norm unity in $[-1, 1]$, $T_n(x)$ has the largest leading coefficient, namely 2^{n-1}.

Thus if $P_n(x)$ is any polynomial of degree n, we can define the minimax polynomial approximation as the calculation of that polynomial $Q(x)$ of degree n-1 or less for which

$$|P_n(x) - Q(x)| \leq \epsilon \qquad -1 \leq x \leq 1$$

for some prescribed $\epsilon \geq 0$.

If $P_n(x)$ is any polynomial of degree n with leading coefficient a_n, then its minimax polynomial approximation of degree n-1 or less on $[-1, 1]$ is

$$Q(x) = P_n(x) - \frac{a_n T_n(x)}{2^{n-1}}$$

Any polynomial of degree n can be uniquely expressed as a linear combination of the Chebyshev polynomials. From Table A we can form Table B in which powers of x are expressed in terms of Chebyshev polynomials. Thus,

TABLE B

$$\begin{aligned}
1 &= T_o \\
x &= T_1 \\
x^2 &= 2^{-1}(T_o + T_2) \\
x^3 &= 2^{-2}(3T_1 + T_3) \\
x^4 &= 2^{-3}(3T_o + 4T_2 + T_4) \\
x^5 &= 2^{-4}(10T_1 + 5T_3 + T_5) \\
x^6 &= 2^{-5}(10T_o + 15T_2 + 6T_4 + T_6) \\
&\vdots
\end{aligned}$$

Therefore, there exist constants C_k such that

$$P_n(x) = C_oT_o(x) + C_1T_1(x) + \cdot\cdot\cdot + C_nT_n(x). \tag{6}$$

We note that the term x^n appears only in $T_n(x)$; therefore, C_n must be equal to $\frac{a_n}{2^{n-1}}$. Thus we can form the minimax polynomial approximation to $P_n(x)$,

$$(7) \qquad Q(x) = C_o + C_1T_1(x) + C_2T_2(x) + \cdot \cdot \cdot - C_{n-1}T_{n-1}(x).$$

Therefore, we see that to obtain the minimax polynomial approximation to a given polynomial $P_n(x)$ we first express $P_n(x)$ as the series (6) of Chebyshev polynomials and then drop the last term.

ECONOMIZATION OF POWER SERIES

It turns out that in many cases one can drop more than one term and still obtain approximations which are "close" to being minimax, which we shall show later in an example illustrating this method of approximating polynomials. In actual practice, we need only retain the terms through some k, $k < n$, in (6) since $|T_j(x)| \leq 1$ for all j. In doing this, the error committed is bounded by

$$\max_{-1 \leq x \leq 1} \left| P_n(x) - \left[C_o + C_1T_1(x) + \cdot \cdot \cdot + C_kT_k\right]\right| \leq |C_{k+1}| + |C_{k+2}| + \cdot \cdot \cdot + |C_n|$$

The procedure above for replacing a polynomial of given degree by one of lower degree is sometimes referred to "economizing a power series". We shall outline this procedure for obtaining an economized polynomial approximation to a function f(x) below.

The economization of a power series has four basic steps.

Step 1. Expand f(x) in a Taylor series valid on the interval

$[-1, 1]$. Truncate this series to obtain a polynomial

$$P_n(x) = a_o + a_1 x + \dots + a_n x^n,$$

which approximates f(x) to within a prescribed tolerance error ϵ for all x in $[-1, 1]$.

Step 2. Expand $P_n(x)$ in a Chebyshev series

$$P_n(x) = C_o + C_1 T_1(x) + \dots + C_n T_n(x)$$

making use of Table B.

Step 3. Retain the first k terms in this series

$$M_k(x) = C_o + C_1 T_1(x) + \dots + C_k T_k(x)$$

choosing k so that the maximum error given by

$$|f(x) - M_k(x)| \leq \epsilon + |C_{k+1}| + \dots + |C_n|$$

is acceptable.

Step 4. Replace $T_j(x)$, (j=0, 1...k) by its polynomial form using Table A and rearrange to obtain the economized polynomial approximation of degree k in standard form,

$$f(x) \approx C_o' + C_1' x - \dots + C_k' x^k$$

If necessary in step 1, make a transformation of independent variables so that the expansion is valid on that interval.

As an example of the above procedure we shall find a polynomial of as low a degree as possible which will yield

the value of e^x for x on $[-1, 1]$ with an error of at most .005.

Step 1. By using the remainder term, $R_n(x) = \frac{(x-a)^n f(\xi)}{n!}$, where a is equal to zero and ξ lies between x and zero in the Taylor series theorem, we find that the first six terms of the Taylor series are required to remain in the prescribed tolerance. Hence,

$$f(x) = e^x \sim 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} = P_5(x)$$

where the maximum absolute error of $P_5(x)$ is less than $\frac{e}{6!} \approx 0.0038$. If we were to omit the $\frac{x^5}{5!}$ term, we would incur the possibility of an additional error, which, when added to that already committed, would exceed the prescribed tolerance.

Step 2. Replacing each power of x in terms of Chebyshev polynomials from Table B, yields

$$P_5(x) = 1.2656250 + 1.1302083T_1 + 0.27083333T_2 + 0.04427083T_3 + 0.0052083T_4 + 0.00052356T_5$$

Step 3. Dropping the last term, we obtain the minimax polynomial approximation to $P_5(x)$:

$$Q_4(x) = 1.2656250 + 1.1302083T_1 + 0.27083333T_2 + 0.04427083T_3 + 0.0052083T_4$$

Now, we need to check our error to see if it lies within the prescribed tolerance level, $\epsilon \leq .005$. Since

$$|P_5(x) - Q_4(x)| \leq 0.00053$$

and $$|e^x - P_5(x)| \leq 0.0038,$$

it follows that $$|e^x - Q_4(x)| \leq 0.0038 + 0.00053 = 0.00433.$$

Hence, the error from using $Q_4(x)$ is less than 0.005 for all x on $[-1, 1]$.

Step 4. Using Table A and replacing T_1, T_2, T_3, T_4 by their polynomial equivalents, and rearranging, we arrive at the economized polynomial:

$$Q_4(x) = 1.000000 + 0.99739581x + 0.50001602x^2 + 0.17708332x^3 + 0.04105064x^4.$$

Letting x take on some values in the interval $[-1, 1]$, we can tabulate the corresponding errors in $P_5(x)$ and $Q_4(x)$:

x	e^x	$e^x - P_5(x)$	$e^x - Q_4(x)$
-1.0	0.36788	0.0012	+0.0007
-0.5	0.60653	0.0000	-0.0002
0	1.0000	0.0000	0.0000
0.5	1.6487	0.0000	0.0003
1.0	2.7183	0.0016	0.0022

Hence, we see that the error in using the polynomials $P_5(x)$ and $Q_4(x)$ are well within our tolerance level of 0.005. The important thing to note is that $Q_4(x)$ is of one degree less than $P_5(x)$ and therefore has fewer terms to tabulate and it still remains in our prescribed tolerance level.

If we can allow an error as large as $\xi \leq .01$, since $|T_n(x)| \leq 1$, and $\left|\frac{x^5}{5!}\right| < .01$ when x lies in the interval $[-1, 1]$, we can neglect the terms in $T_4(x)$ and $T_5(x)$. Hence, in so doing, we introduce an additional error of at most $\frac{11}{1920} \approx 0.0057$. Thus, we may write

$$f(x) = e^x \approx \frac{81}{64}T_0(x) + \frac{219}{192}T_1(x) + \frac{13}{48}T_2(x) + \frac{17}{348}T_3(x)$$

where the maximum error of about 0.0095 is within error tolerance set. When we convert the above back into the economized polynomial in x, we obtain

$$e^x \approx \frac{1}{384}(382 + 383x + 208x^2 + 68x^3)$$

where the error is not more than .01 throughout the interval $[-1, 1]$.

For many functions, especially those with slowly converging power series, the telescoping effect can be quite dramatic. C. Hastings [(2), p. 106] shows that the polynomial

$$Q_6(x) = 0.99990167x - 0.49787544x^2 + 0.31765005x^3 - 0.19376149x^4 + 0.08556927x^5 - 0.01833851x^6$$

approximates ln(1+x) to an accuracy of 0.0000015 for all x on the interval $[0, 1]$. The Taylor series for ln(1+x) converges so slowly that many hundreds of terms would be required for this same accuracy.

In actual practice, we do not need to carry out all of the above computations. Consider again the truncated series

$$(8) \qquad f_n(x) = \sum_{k=0}^{n} a_k x^k \qquad -1 \le x \le 1.$$

Suppose that the error obtained by this approximation is R_n, and that the prescribed maximum error is ε.

If we write

$$(9) \qquad f(x) = \tfrac{1}{2}b_0 + \sum_{k=1}^{n} b_k T_k(x)$$

we have

$$b_n = \frac{a_n}{2^{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}}{2^{n-2}}$$

$$b_{n-2} = \frac{1}{2^{n-1}} \left[\binom{n}{1} a_n + 2^2 a_{n-2}\right]$$

$$b_{n-3} = \frac{1}{2^{n-2}} \left[\binom{n-1}{1} a_{n-1} + 2^2 a_{n-3}\right]$$

$$b_{n-4} = \frac{1}{2^{n-1}} \left[\binom{n}{2} a_n + 2^2 \binom{n-2}{1} a_{n-2} + 2^4 a_{n-4} \right]$$

$$\vdots$$

We now define

$$B_k = \sum_{j=n-1-k}^{n} |b_j|$$

Suppose $B_k \leq \varepsilon - R_n$, but $B_{k+1} > \varepsilon - R_n$, then we simply substract the last k terms of the series (9), from the series (8), and by so doing obtain an approximation with maximum error

$$R_{n-k} \leq R_n + B_k.$$

We will now rework the previous example employing these ideas. We have that $\varepsilon = 0.01$, $R_5 = 0.0038$, and $|a_k| = \frac{1}{k!}$

Then

$$b_5 = \frac{1}{16} \cdot \frac{1}{5!} \qquad B_1 = 0.000521$$

$$b_4 = \frac{1}{8 \cdot 4!} \qquad B_2 = 0.005792$$

$$b_3 = \frac{1}{16} \left[5 \cdot \frac{1}{5!} + 4 \cdot \frac{1}{3!} \right] \qquad B_3 = 0.044$$

Now $\varepsilon - R_5 = 0.0062$. Since $B_2 < 0.0062$ and $B_3 > 0.0062$ our approximation is

$$f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \left[\frac{1}{16 \cdot 5!} T_5(x) + \frac{1}{8 \cdot 4!} T_4(x) \right]$$

with an error R_3 not exceeding $R_5 + B_2 = 0.0095$. Performing the indicated substraction we obtain, as before,

$$e^x \approx \frac{1}{384}\left[382 + 383x + 208x^2 + 68x^3\right]$$

with the maximum error R_3 with the prescribed tolerance.

As the exponential function approximation indicates, truncated power series have the property that the maximum error occurs at the end points of the interval of interest. If we approximate the function $f(x) = \sin\frac{1}{2}\pi x$ for $-1 \leq x \leq 1$, the truncated Taylor series expansion about $x = 0$ is

$$(10) \quad \sin\tfrac{1}{2}\pi x \approx 1.5708x - 0.6460x^3 + 0.0797x^5 - 0.0047x^7.$$

Now economizing this series approximation by use of Chebyshev polynomials we obtain

$$(11) \quad \sin\tfrac{1}{2}\pi x \approx 1.1336T_1(x) - 0.1381T_3(x) + 0.0045T_5(x).$$

The representation of $\sin\frac{1}{2}$ x is thereby achieved with the storage of only three numbers, the coefficients of $T_1(x)$, $T_3(x)$, and $T_5(x)$; the right of (11) may then be readily evaluated.

The desk-machine user is seldom attracted by the compactness of (11); he usually insists on seeing the function values and differences in order to ascertain at a glance the behavior of the function, and to obtain a reliable check against isolated

computing errors. However, for an automatic computer, which is much less prone to isolated errors, the Chebyshev series representation is preferable.

In this particular example, we note that the Taylor series expansion has one more term than (11); if the Taylor series is truncated after the third term, its maximum error is larger than that for series (11).

This is a simple example of the general property that, in a given finite range, an approximation in Chebyshev series of prescribed degree represents a function of a real variable more accurately than a truncated Taylor series of the same degree. In the special case where the function happens to be a polynomial of the required degree, the Chebyshev and Taylor representations are equally accurate, each being a rearrangement of the other. Moreover, any function which can be represented by an orthodox single-entry table can be represented by a single Chebyshev series, whereas a Taylor series valid over the whole tabular range may not exist.

We may summarize by restating that a truncated Chebyshev series is normally a good approximation to the best polynomial representation in the sense of Chebyshev's criteria. However, it transpires that in practical applications the truncated Chebyshev series is usually very close to the best possible polynomial; the refinements necessary to improve it are seldom worthwhile.

CONCLUSION

In conclusion, in this report we have defined the Chebyshev polynomials by means of the relation $T_n(x) = \cos(n \arccos x)$. We have shown how they satisfy certain differential equations arising from the solution of the Sturm-Liouville problem. Also, we have shown that they can be obtained from a particular generating function, a Rodrigues-type formula, and a three term recurrence relation. We have shown that Chebyshev polynomials are orthogonal over $[-1, 1]$ with respect to a certain weight factor. They also possess the useful property that $|T_n(x)| \leq 1$ and $T_n(x)$ obtains its extreme values alternately of $(-1)^k$ at $n + 1$ points over the interval $[-1, 1]$.

We have utilized the Fourier Series property that the Fourier-Chebyshev series minimizes the average square error. An example is given to illustrate this idea. One of the most important properties discussed in this report leads to the minimax principle, along with the proof and an example to fit the principle. Then from the minimax principle, the economization of power series was developed and it was shown that certain functions can be evaluated numerically by this economizing process. The study of Chebyshev polynomials leads to many interesting mathematical problems.

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CHEBYSHEV APPROXIMATIONS

by

PHILLIP F. UNRUH

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The name "Chebyshev polynomials" is in honor of the Russian mathematician P. F. Chebyshev (1821-1894). Most of his contributions to mathematics were in the area of number theory.

In this report the basic definition of the Chebyshev polynomials we have employed is given by $T_n(x) = \cos(n \arccos x)$. We have shown how they satisfy certain differential equations arising from the solution of a specific Sturm-Liouville problem. Also, we have shown that they can be obtained from a particular generating function, a Rodrigues-type formula, and a three term recurrence relation. We have shown that Chebyshev polynomials are orthogonal over $[-1, 1]$ with respect to a certain weight factor. They also possess the useful property that $|T_n(x)| \leq 1$ and that $T_n(x)$ obtains its extreme values of ± 1 at n+1 points over the interval $[-1, 1]$.

We have utilized the Fourier Series property that the Fourier-Chebyshev series minimizes the average square error. An example is given to illustrate this idea. One of the most important properties discussed in this report leads to the minimax principle, along with the proof and an example illustrating the principle. Then from the minimax principle, the economization of power series was developed and it was shown that certain functions can be evaluated efficiently, numerically, by this economizing process.